

Method for the Self-Consistent Determination of Regge Pole Parameters*

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(Received 7 November 1963)

A method is suggested for approximately bootstrapping Regge trajectories, thereby avoiding the cutoff problems of the usual bootstrap calculation. The method is based on dispersion relations for Regge trajectories and on unitarity applied at $l=\alpha$. Successively more realistic approximations are described which bring in more information on the potential, and more trajectories. The approximate Regge parameters are guaranteed to have the desired threshold and asymptotic properties.

I. INTRODUCTION

ONE basic drawback of previous bootstrap calculations has been the presence of arbitrary parameters (cutoffs and the like), which were necessary in order to describe, at least crudely, the unknown high-energy region. It is now believed that the high-energy behavior of scattering amplitudes is controlled by Regge trajectories corresponding to particles in the crossed channels. Therefore, if a method could be developed to carry out a bootstrap calculation of an entire Regge trajectory, rather than simply of one point on that trajectory, the high-energy region would itself be capable of a self-consistent determination.

Nonrelativistic potential scattering is the natural breeding ground for studying the properties of the Regge poles, so we shall first search for a method of computing Regge trajectories here. It is, of course, essential to phrase the method in a way not tied directly to the Schrödinger equation, but to express it in language easily extendible to the relativistic case. Perhaps the most obvious way to do this is to base the method on the dispersion relations known to be satisfied by the trajectories and their residues in potential theory. The dispersion relations may then be supplemented by unitarity to obtain a system of integral equations which couple all trajectories together. One may then construct approximations of (more or less) practicable value by neglecting all but a few trajectories.

The simplest approximation is to neglect all but one trajectory, and this case is discussed in Sec. II. The equations obtained there have been used by Cheng and Sharp¹ in a slightly modified form to compute approximate Regge trajectories, but their technique is not suitable for applications to bootstrapping because of the presence of undetermined parameters. In Sec. III

we generalize the method in two stages, based on the Khuri representation² for Regge poles. In Sec. IV we generalize the method further, by discussing the inclusion of several Regge poles in the coupled equations. It is interesting to see how details of the potential come in through the positions of the zeros of the residue functions. Section V describes the relativistic generalization of the method and its applicability to the bootstrap. Finally, in Sec. VI, a brief comparison is made with Chew's suggestions³ for bootstrapping an entire Regge trajectory.

II. FIRST VERSION OF THE SINGLE TRAJECTORY EQUATIONS

Our basic idea, as we have stated in the Introduction, is to construct a method by which we can obtain, from the bootstrap principle, the entire Regge trajectory associated with a particle. The method we shall suggest is based upon the use of the dispersion relations satisfied by the trajectories and their associated residue functions, and it will be helpful to discuss it first within the context of the potential theory. Here, of course, there is no question of bootstrapping a trajectory; we can only carry out the calculation of trajectories resulting from a given input potential. Nevertheless, there is a very close parallel between this and the final proposal for a relativistic bootstrap. Furthermore, the potential theory case, where exact solutions for the trajectories exist, gives us an opportunity to measure the accuracy of our approximation.

It has by now been well established⁴ that the leading Regge trajectories $\alpha_n(s)$ in potential theory satisfy

² N. N. Khuri, Phys. Rev. **130**, 429 (1963).

³ G. F. Chew, Phys. Rev. **129**, 2363 (1963).

⁴ A. Barut and D. Zwanziger, Phys. Rev. **127**, 974 (1962); and H. Cheng, Phys. Rev. **130**, 1283 (1963). Trajectories other than the leading ones often cross and develop left-hand cuts. In that case, (2.1) is not satisfied. This case will be discussed further in Sec. IV.

* Work supported in part by the Alfred P. Sloan Foundation and the U. S. Atomic Energy Commission.

¹ H. Cheng and D. Sharp, Phys. Rev. **132**, 1854 (1963).

dispersion relations in the energy s of the form

$$\alpha_n(s) = -n + \frac{1}{\pi} \int_0^\infty \frac{ds'}{s' - s - i\epsilon} \text{Im}\alpha_n(s'). \quad (2.1)$$

It is also true that $b_n(s)$, defined by⁵

$$b_n(s) = \beta_n(s)/s^{\alpha_n(s)}, \quad (2.2)$$

where $\beta_n(s)$ is the residue of the partial-wave amplitude associated with the pole at $l = \alpha_n(s)$, is an analytic function of s with only a right-hand cut. For a superposition of Yukawa potentials,

$$V(r) = \int_m^\infty d\mu \frac{e^{-\mu r}}{r} \sigma(\mu),$$

$b_n(s)$ has the asymptotic behavior

$$b_n(s) \rightarrow (g^2/2)s^{n-1} \quad \text{as } s \rightarrow \infty, \quad (2.3)$$

where

$$g^2 = - \int_m^\infty \sigma(\mu) d\mu.$$

We may now use the analyticity of $b_n(s)$, this asymptotic behavior, and the fact, proved in Sec. IV, that $b_n(s)$ has exactly $n-1$ zeros, to derive the following useful representation for $\beta_n(s)$.

We consider the function

$$\phi(s) = \ln(b_n(s) / \prod_{i=1}^{n-1} (s - s_i)),$$

where the s_i are the locations of the zeros of $b_n(s)$. Note that all the s_i are real.⁶ This function is analytic with a right-hand cut starting at $s=0$, has no singularities, and asymptotically becomes $\ln(g^2/2)$. Therefore,

$$\phi(s) = \ln(g^2/2) + \frac{1}{\pi} \int_0^\infty \frac{\text{Im}\phi(s')}{s' - s - i\epsilon} ds'. \quad (2.4)$$

Now $\text{Im}\phi(s')$ is just the phase of $b_n(s')$, which, by Eq. (2.2), is the same as the phase of $\beta_n(s')$ minus $\text{Im}\alpha_n(s') \times \ln s'$. Hence, we get

$$\beta_n(s) = \frac{g^2}{2} \prod_{i=1}^{n-1} (s - s_i) \exp \left(\frac{1}{\pi} \int_0^\infty \left[\tan^{-1} \frac{\text{Im}\beta_n(s')}{\text{Re}\beta_n(s')} - \text{Im}\alpha_n(s') \log s' \right] \frac{ds'}{s' - s} + \alpha_n(s) \ln s \right). \quad (2.5)$$

⁵ We choose the particle mass equal to $\frac{1}{2}$ so that $s = q^2$, where q is the particle momentum.

⁶ As will be made evident in Sec. IV, the s_i are real and negative for normal trajectories; i.e., for trajectories with only the usual right-hand cut in energy starting at $s=0$. For abnormal ones, that is, trajectories which may for example cross and therefore have additional branch points, the s_i may be complex. The derivation given here must then be modified in an obvious way.

With the use of the dispersion relation for $\alpha_n(s)$, Eq. (2.5) can finally be transformed into

$$\beta_n(s) = \frac{g^2}{2s} \prod_{i=1}^{n-1} \left(\frac{s - s_i}{s} \right) \exp \left(\frac{1}{\pi} \int_0^\infty \frac{ds'}{s' - s} \times \left[\text{Im}\alpha_n(s') \ln \frac{s}{s'} + \tan^{-1} \frac{\text{Im}\beta_n(s')}{\text{Re}\beta_n(s')} \right] \right). \quad (2.6)$$

This equation, which, we remain the reader, is exact, will be very useful in what follows.

The dispersion relations by themselves do not, of course, constitute a dynamical scheme which may be used for calculating anything. They must be supplemented by some information about the imaginary parts. As is the case with the usual dispersion relation for scattering amplitudes, this information is supplied by unitarity.

The original representation derived by Regge,⁷ expressing a scattering amplitude in terms of Regge poles, allows one to write for the partial-wave amplitude that

$$A(s, l) = \sum_n \frac{\beta_n(s)}{l - \alpha_n(s)} + B(s, l). \quad (2.7)$$

The sum is over all Regge poles; $B(s, l)$ is the contribution of the mysterious "background integral." The partial-wave amplitude satisfies the unitarity condition

$$[A(s, l) - A(s, l^*)^*] / 2i = (s)^{1/2} A(s, l) A(s, l^*)^*. \quad (2.8)$$

Applying this at $l = \alpha$ gives us the equations

$$\frac{1}{2i(s)^{1/2}} = \sum_n \frac{\beta_n^*(s)}{\alpha_n(s) - \alpha_n^*(s)} + B(s, \alpha_n^*(s))^*, \quad m = 1, 2, \dots \quad (2.9)$$

Equations (2.9) are an infinite set of relations connecting the imaginary parts of the α 's and β 's, given the function B . In conjunction with the dispersion relations, they provide us with an infinite set of coupled integral equations for the trajectories. As a practical matter, solving these equations is manifestly out of the question, and some approximation is called for. One could, for example, make the set of equations finite by discarding all but a finite number, say N , of poles and residues. Then the sum in Eq. (2.9) runs up to N , m goes from 1 to N , and we have N complex or $2N$ real algebraic equations for the $2N$ real quantities $\text{Im}\beta$ and $\text{Im}\alpha$.

To begin with, let us suppose we take $N=1$ and see what happens. There is no reason to believe this will be a terrible realistic approximation; one may argue that if $\text{Im}\alpha$ is small, as it presumably will be at low and high energies, and for weak potentials, then the term in the sum with $n=m$ will dominate in Eq. (2.9). On this basis we may therefore also neglect B , and Eq. (2.9) reduces to

$$\text{Im}\alpha = (s)^{1/2} \beta \quad \text{and} \quad \text{Im}\beta = 0. \quad (2.10)$$

⁷ T. Regge, Nuovo Cimento 14, 951 (1959).

(We drop the subscript 1 on α_1 and β_1 .) From Eq. (2.6) we find (remember β_1 has no zeros)

$$\text{Im}\alpha(s) = \frac{g^2}{2(s)^{1/2}} \exp \left[\frac{1}{\pi} \int_0^\infty \frac{ds'}{s'-s} \text{Im}\alpha(s') \log s/s' \right]. \quad (2.11)$$

This integral equation for $\text{Im}\alpha$ has several pleasing features:

(i) It gives the correct asymptotic form for $\text{Im}\alpha$; viz: $\text{Im}\alpha(s) \rightarrow g^2/2(s)^{1/2}$ as $s \rightarrow \infty$.

(ii) As $s \rightarrow 0$, it gives $\text{Im}\alpha(s) \rightarrow s^{\alpha(0)+1/2}$, which is also correct.

(iii) If the scattering amplitude itself is approximated by just one pole, whose trajectory satisfies Eq. (2.10), so that

$$A(s, l) = \beta/(l - \alpha),$$

then the unitarity condition (2.8) is automatically satisfied.

It has, in addition, several unpleasing features:

(i) The range of the potential does not occur in the equation.

(ii) The coupling constant scales out, in that $\text{Im}\alpha(s)$ is a function only of s/g^4 .

The solution of Eq. (2.11) is thus some sort of "universal" Regge trajectory, which is essentially completely independent of the potential.⁸ This is obviously physical nonsense, and it is not hard to find the source of the difficulty. In the "one pole" approximation used in obtaining Eq. (2.11), we have discarded the "background term" $B(s, l)$. Yet, a glance at Eq. (2.1) shows us that the entire left-hand cut in energy of the partial-wave amplitude $A(s, l)$, and thereby all information on the forces in the problem, is contained in $B(s, l)$. In order to make sense of an approach such as this, then, it is essential to incorporate the "background term," or part of it at least, into a practicable approximation scheme. A way to do this will be suggested in the following section.

III. KHURI MODIFICATION TO SINGLE TRAJECTORY EQUATIONS

An alternative form of the Regge representation of a scattering amplitude to the one which led to Eq. (2.9)

⁸ H. Cheng and D. Sharp, Phys. Rev. **132**, 1854 (1963) have suggested replacing the unsubtracted dispersion relations for α and b by subtracted ones, with the subtraction constants α_0 and b_0 , and the subtraction points s_0 , considered to be parameters which may be taken from an exact solution or from experiment. This modifies Eq. (2.11) to

$$\text{Im}\alpha = b_0 s^{\alpha_0+1/2} \exp \frac{s-s_0}{\pi} \int_0^\infty \frac{ds'}{(s'-s_0)(s'-s)} \text{Im}\alpha(s') \ln s/s'.$$

Some information about the forces is thus introduced through α_0 , b_0 , and s_0 . They find the approximate trajectories calculated in this way to be surprisingly accurate at low energies. Asymptotically, of course, they become seriously wrong. For our purposes, however, the presence of unspecified parameters renders this approach useless.

has been proposed by Khuri.² It has the virtue of, under certain plausible conditions, incorporating the entire background integral contribution into the contribution of the Regge poles. It looks like this:

$$A(s, l) = \sum_n \frac{\beta_n(s)}{l - \alpha_n(s)} \exp[-(l - \alpha_n(s))\xi], \quad (3.1)$$

where $\xi = \cosh^{-1}(1 + m^2/2s)$ and where m is the potential range; that is, the potential is assumed to be expressible as

$$V(r) = \int_m^\infty d\mu (e^{-\mu r}/r) \sigma(\mu) \quad (3.2)$$

with

$$g^2 = - \int_0^\infty d\mu \sigma(\mu).$$

Using this form for the partial-wave amplitude, the analog of Eq. (2.9) is simply

$$\frac{1}{2i(s)^{1/2}} = \sum_n \frac{\beta_n^*(s)}{\alpha_m(s) - \alpha_n^*(s)} \times \exp[-(\alpha_m(s) - \alpha_n^*(s))\xi], \quad (3.3)$$

and in the one pole approximation, this leads to an integral equation for $\text{Im}\alpha$ slightly different from (2.11). The phase of $\beta(s)$ is now just minus $2\xi(s) \text{Im}\alpha(s)$, so by using (2.6) again, we obtain

$$\text{Im}\alpha(s) = \frac{g^2}{2(s)^{1/2}} \exp \left[\frac{1}{\pi} \int_0^\infty \frac{ds'}{s'-s} \text{Im}\alpha(s') \times (\ln s/s' - 2\xi(s')) \right]. \quad (3.4)$$

The principal value is necessary on the integral, in contrast to Eq. (2.11), since the ξ term does not vanish at $s = s'$.

Equation (3.4) retains the first two of the pleasing features of Eq. (2.11). It is still true that $\text{Im}\alpha(s)$ asymptotically approaches $g^2/2(s)^{1/2}$; it is also still true that near threshold $\text{Im}\alpha(s)$ behaves like $s^{\alpha(0)+1/2}$. On the negative side, the unitarity condition is no longer satisfied by the "one pole" approximation to the scattering amplitude, except, of course, at $l = \alpha$.

Numerical solutions to Eq. (3.4) have been obtained, and compared with the exact solutions of Amadzadeh *et al.*⁹ for a single Yukawa potential. The general shape of the exact results are reproduced by the approximation, but in magnitude the exact $\text{Im}\alpha$ lies considerably above the approximate one. The difference in magnitude is about a factor of two for a coupling of $g^2 = 1.8$; a factor of three to four for $g^2 = 5$.

In this form, the approximate trajectory now does depend on the potential strength and range in a non-

⁹ A. Amadzadeh, P. Burke, and C. Tate, Phys. Rev. **131**, 1315 (1963).

trivial way; it does not, however, depend on more subtle details. One way to bring in more features of the potential, through the inclusion of more Regge poles in the approximation, will be described in Sec. IV. Another way to do the same thing, by explicitly extracting the Born-approximation term so that it stands out by itself in the expression for the partial-wave amplitude, will be described next.

For a single Yukawa potential, the Khuri representation is perhaps still not all we could desire. The original motivation for constructing the Khuri form was to arrange that each Regge pole term by itself had a cut in the $\cos\theta$ plane (θ is the scattering angle) from $1+m^2/2s$ to ∞ , and thus eliminate the miraculous calculation of cuts between 1 and $1+m^2/2s$ required among the Regge poles and background integral of the old representation. For a superposition of Yukawa potentials, as in Eq. (3.2), where the scattering amplitude actually does have a cut all the way from $\cos\theta = 1+m^2/2s$ to ∞ , the Khuri form is thus eminently suitable. However, for a single Yukawa, the cut really begins at $\cos\theta = 1+4m^2/2s$; there is only an isolated pole at $\cos\theta = 1+m^2/2s$. It would therefore be desirable to have available a modified Khuri form in which each Regge pole term has only these singularities in $\cos\theta$.

Such a form may easily be constructed. We have only to consider the difference of the scattering amplitude and the Born approximation, and apply Khuri's arguments to this function. Let

$$\bar{A}(s, \cos\theta) = A(s, \cos\theta) - A_B(s, \cos\theta), \quad (3.5)$$

where the Born approximation is

$$A_B(s, \cos\theta) = \frac{g^2}{2s} \frac{1}{(1+m^2/2s) - \cos\theta} \quad (3.6)$$

and contains the pole at $\cos\theta = 1+m^2/2s$. Since \bar{A} satisfies a dispersion relation in $\cos\theta$, with $1+4m^2/2s$ as the lower limit on the dispersion integral, it is reasonable to expect that the partial-wave amplitude is bounded as follows:

$$\bar{A}(s, l) \lesssim (l)^{-1/2} \exp(-\text{Re}l\bar{\xi}), \quad (3.7)$$

where $\bar{\xi} = \cosh^{-1}(1+4m^2/2s)$. Blindly following Khuri's arguments then yields trivially the representation¹⁰

$$A(s, l) = \sum_{l-\alpha_n} \frac{\beta_n}{\exp[-(l-\alpha_n)\bar{\xi}]} + \frac{g^2}{2s} Q_l(1+m^2/2s) - \frac{g^2}{2s} \sum_{n=1}^{\infty} \frac{P_{n-1}(1+m^2/2s)}{l+n} \exp[-(l+n)\bar{\xi}]. \quad (3.8)$$

This procedure has provided everything we asked for, in that in this way we have constructed Regge pole

terms each of which has a pole at $\cos\theta = 1+m^2/2s$ and a cut starting at $\cos\theta = 1+4m^2/2s$. It is, of course, not true that each Regge term in this form of the representation satisfies the Mandelstam representation with the correct boundary on the double spectral function. The integration region here is just $s > 0$, $t > 4\mu^2$.

To obtain the "one-pole approximation" following from this modified Khuri representation, we drop all of the sums over Regge poles in Eq. (3.8) except that containing the leading trajectory. In addition, we shall throw away the sum $n=2$ to ∞ in the last term of (3.8), retaining only the $n=1$ term. The partial-wave amplitude then contains a moving pole at $l=\alpha$, the largest Regge pole, and in addition has fixed poles at $l=-2, -3, -4, \dots$, etc., coming from the Q_l term. The pole at $l=-1$ in Q_l is cancelled by the piece we have kept of the last term in Eq. (3.8). In this way, the approximation consists effectively in fixing all the trajectories except the largest at their $s=\infty$ values, and allowing only the largest one to move as s is increased from $-\infty$. If we did not discard all but the $n=1$ part of the last term in (3.8), but, say, kept all of it, our approximate partial-wave amplitude would have only a single pole in the l plane, namely, that at $l=\alpha$. By retaining only the $n=1$ piece, then, we have kept at least some effect of the lower Regge poles in the approximation. Approximately, then, we may write

$$A(s, l) = \frac{\beta}{l-\alpha} \exp[-(l-\alpha)\bar{\xi}] + \frac{g^2}{2s} Q_l(1+m^2/2s) - \frac{g^2}{2s} \frac{1}{l+1} \exp[-(l+1)\bar{\xi}], \quad (3.9)$$

and consequently, by using the unitarity condition at $l=\alpha$, we have

$$\beta = \text{Im}\alpha \exp(-2i \text{Im}\alpha \bar{\xi}) \left(\frac{1}{(s)^{1/2}} + i \frac{g^2}{s} Q_{\alpha^*}(1+m^2/2s) - i \frac{g^2}{s} \frac{1}{\alpha^*+1} \exp[-(\alpha^*+1)\bar{\xi}] \right). \quad (3.10)$$

From this equation it is straightforward, though messy, to calculate the phase of β in terms of α , and therefore, through the dispersion relation for α , in terms of $\text{Im}\alpha$ alone. This phase and β itself may then be inserted into Eq. (2.6) to get a frightening looking and remarkably nonlinear integral equation for $\text{Im}\alpha$. We shall not explicitly write this equation.

Even though the integral equation for $\text{Im}\alpha$ is terribly complicated, it is not out of the question to obtain numerical solutions of it, and attempts to do this are in progress, again with the intent of comparing with the exact results for a Yukawa potential.

It is not difficult to see, in spite of the complexity of the integral equation, that the solution will have the correct threshold and asymptotic properties. As $s \rightarrow \infty$,

¹⁰ A. Ahmadzadeh, Lawrence Radiation Laboratory Report UCRL-10929 (unpublished), has also written down Eq. (3.8). His numerical analysis of it encourages the belief that it will be more useful than the original Khuri representation.

for example, Eq. (3.10) becomes

$$(s)^{1/2}\beta = \text{Im}\alpha, \quad (3.11)$$

and, from Eq. (2.6), we see that consequently $\text{Im}\alpha \rightarrow g^2/2(s)^{1/2}$. As $s \rightarrow 0$, Eq. (3.10) also reduces to (3.11) if $\alpha(0) > -\frac{1}{2}$, and Eq. (2.6) shows us that $\text{Im}\alpha \sim s^{\alpha(0)+1/2}$. Furthermore, the correct analyticity properties of α are, of course, guaranteed, since the whole approximation is based on use of the dispersion relation for α . Unfortunately, $A(s, l)$ as expressed by Eq. (3.9) does not satisfy unitarity except at $l = \alpha$.

The entire Born-approximation term appears explicitly in Eq. (3.9), so that the result depends on the full details of the potential. Furthermore, through the presence of the Born term, the correct asymptotic behavior of $A(s, l)$ in l as well as in s is guaranteed.

For these reasons, it may be desirable to use the modified Khuri representation, and the approximations generated from it (appropriately corrected, of course) in the case of a superposition of Yukawa potentials, even though the original motivation for constructing the representation was present only for a single Yukawa potential.

IV. COUPLED TRAJECTORIES

To derive the number of zeros¹¹ in b_n for normal trajectories (those with no left-hand cut), we need only compare the known asymptotic limit,

$$b_n(s) \rightarrow (g^2/2)s^{n-1} \quad \text{as } s \rightarrow \infty,$$

for potential theory with the representation

$$b_n(s) = c \prod_{i=1}^N (s - s_i) \exp \left[\frac{1}{\pi} \int_0^\infty \frac{\phi(s') ds'}{s' - s} \right] \quad (4.1)$$

for a function analytic in s with only a right-hand cut. The phase ϕ of b_n is known to vanish both at $s=0$ and $s=\infty$. Therefore, the exponential in (4.1) approaches a constant asymptotically, and $b_n \sim s^N$. By comparison with the known asymptotic form of b_n , one finds

$$N = n - 1. \quad (4.2)$$

Thus, the first trajectory has no zeros, the second has one zero, and so forth. Cheng and Sharp¹¹ have also considered trajectories which have a left-hand cut and find that the asymptotic form is still valid and the phase still vanishes asymptotically; Eq. (4.2) holds in general.

In the case of potential scattering, the zeros of $b_n(s)$ are roots of finite polynomials, with coefficients easily determined in terms of moments of the potential. For a normal trajectory, the zeros of $b_n(s)$ are the zeros of the residues $\beta_n(s)$.

If the S matrix is expressed by

$$S(s, l) = e^{i\pi l} D(s^-, l) / D(s^+, l), \quad (4.3)$$

¹¹ H. Cheng and D. Sharp (Ref. 1) have already discussed this topic. We reproduce the discussion here for completeness.

then the conditions for a zero of $\beta_n(s)$ become

$$D(s^+, l) = D(s^-, l) = 0. \quad (4.4)$$

This is the condition for indeterminacy points.^{12,13} The condition can be met only at negative integer and half-integer l values, as can be seen from the equation

$$D(s^+, l)D(s^-, -l-1) - D(s^-, l)D(s^+, -l-1) = -(2i/\pi)s^+ \sin 2\pi l. \quad (4.5)$$

In the right-hand l plane, the wave function which goes as r^{l+1} at the origin is regular, and therefore $D(s^+, l)$ and $D(s^-, l)$ cannot vanish simultaneously since this would make the wave function vanish identically. In the left-hand l plane, this restriction does not apply. Thus, at negative integer or half-integer l values, whenever $D(s^+, l)$ vanishes, either $D(s^+, -l-1)$ vanishes¹⁴ and some trajectory passes through the corresponding positive l , or $D(s^-, l)$ vanishes, which defines an indeterminacy point, and which is a necessary and sufficient condition that this $l-s$ combination is some $\alpha_n(s)$ with the corresponding $\beta_n(s)$ vanishing.

The calculation of these points is straightforward.^{10,12,13} For the potential (3.2), we define the moments

$$v_n = \frac{(-1)^n}{n!} \int_m^\infty \mu^n \sigma(\mu) d\mu, \quad v_0 = -g^2. \quad (4.6)$$

The power series solution for the wave function which goes like r^{l+1} near the origin is given by the recursion relation

$$\psi(l, s, r) = r^{l+1} \sum_{p=0}^\infty a_p r^p, \quad (4.7)$$

$$a_p = \frac{-sa_{p-2} + \sum_{n=0}^{p-1} v_n a_{p-1-n}}{p[p+2l+1]}, \quad a_0 = 1. \quad (4.8)$$

In order to avoid poles in $D(s, l)$ at all negative integer and half-integer l values, we normalize by putting

$$\bar{a}_p = a_p / \Gamma(2l+2). \quad (4.9)$$

Due to this normalization, as a negative integer or half-integer l is approached, the first $2l+1$ coefficients will vanish. The coefficient becomes

$$\begin{aligned} \bar{a}_{-2l-1} &= \lim_{\epsilon \rightarrow 0} \frac{-sa_{-2l-3} + \sum_{n=0}^{-2l-2} v_n \bar{a}_{-2l-2-n}}{-[2l+1]\epsilon \Gamma(2l+2+\epsilon)} \\ &= e^{i\pi(2l+2)} \frac{(-2l-2)!}{-(2l+1)} \left[-sa_{-2l-3} + \sum_{n=0}^{-2l-2} v_n \bar{a}_{-2l-2-n} \right] \\ & \quad l = -1 - N/2, \quad N = 0, 1, \dots \quad (4.10) \end{aligned}$$

¹² P. Kaus, Nuovo Cimento **29**, 598 (1963).

¹³ H. Bethe (to be published).

¹⁴ S. Mandelstam, Ann. Phys. **19**, 254 (1962).

In general, the coefficient $\bar{a}_{-(2l+1)}$ is now finite and the solution starts with r^{-l} and is therefore proportional to the solution at $-l-1$ starting as r^{l+1} . But for a finite number of energies s_i , the coefficient $\bar{a}_{-(2l+1)}$ will vanish. In this case, all subsequent \bar{a}_p vanish and consequently so do the functions $D(l, s_i^+)$ and $D(l, s_i^-)$, which are the coefficients of the exponentials of the large r asymptotic form of the wave function.

The condition for indeterminacy points is now given by

$$-s_i a_{-2l-3} + \sum_{n=0}^{-2l-2} v_n a_{-2l-2-n} = 0, \quad l = -1 - N/2, \quad N = 0, 1, \dots, \quad (4.11)$$

where a_p is given by the recursion relation (4.8).

For example, one obtains for the first few indeterminacy points:

$$\begin{aligned} \alpha_n(s_1) &= -\frac{3}{2} & s_1 &= -v_0^2 + v_1 \\ \alpha_n(s_2) &= -2 & s_2 &= -(\frac{1}{4})v_0^2 + v_1 - v_2/v_0 \\ \alpha_n(s_3) &= \alpha_n(s_4) = -\frac{5}{2} & s_{3(4)} &= (1/18)[-10v_0^2 + 18v_1 \\ & & & \quad - (+)4(4v_0^2 - 54v_0v_2 \\ & & & \quad + 81v_3^{1/2})], \text{ etc.} \end{aligned} \quad (4.12)$$

In the case of a single Yukawa potential, the indeterminacy points reduce to

$$\begin{aligned} \alpha_n(s_1) &= -\frac{3}{2} & s_1 &= g^2 m - g^4 \\ \alpha_n(s_2) &= -2 & s_2 &= -(\frac{1}{2})m^2 + g^2 m - (\frac{1}{4})g^4 \\ \alpha_n(s_3) &= \alpha_n(s_4) = -\frac{5}{2} & s_{3(4)} &= g^2 m - (5/9)g^4 - (+)(2/9) \\ & & & \quad \times [(27/2)g^2 m^3 - 27g^4 m^2 \\ & & & \quad + 4g^8]^{1/2}, \text{ etc.} \end{aligned} \quad (4.13)$$

When all the relevant trajectories are normal, which is the case for strong attractive coupling, the top trajectory does not use any of these points, $\alpha_2(s)$ uses s_1 to cross $l = -\frac{3}{2}$, and $\alpha_3(s)$ uses s_3 and s_2 to cross $l = -\frac{5}{2}$ and -2 , respectively. To cross higher negative-integer and half-integer l , the trajectories use the zeros of $D(-l-1, s^+)$, i.e., the Mandelstam symmetry.¹⁴ Thus, we have the following information from the indeterminacy points:

$$\begin{aligned} \alpha_2(s_1) &= -\frac{3}{2} & b_2(s_1) &= 0 \\ \alpha_3(s_3) &= -\frac{5}{2} & b_3(s_3) &= 0 \\ \alpha_3(s_2) &= -2 & b_3(s_2) &= 0, \text{ etc.} \end{aligned} \quad (4.14)$$

This process can be continued, the s_i appearing as roots of polynomials of degree $-\lceil \alpha_n + 1 \rceil$ for α_n a negative integer and $-\lceil \alpha_n + \frac{1}{2} \rceil$ for α_n a negative half-integer. As trajectories with larger n are computed and coupled to the others, more moments of the potentials are needed for the computation of the s_i . It is in this way that the detailed structure of the potential eventually gets into the problem.

In Sec. II we derived an exact expression for the residue function $\beta_n(s)$ [Eq. (2.6)], which expressed it in terms of its phase, its zeros, and $\text{Im}\alpha_n(s)$. In Secs. II

and III, we always confined ourselves to the largest trajectory; however, with explicit expressions for the zeros s_i , we are now in a position to extend our discussion to include additional trajectories.

For the second and third trajectories, for example, we have

$$\beta_2(s) = \frac{g^2}{2s} \left(1 - \frac{s_1}{s}\right) \exp \left\{ \frac{1}{\pi} \int_0^\infty \frac{ds'}{s' - s} \left[\tan^{-1} \frac{\text{Im}\beta_2(s')}{\text{Re}\beta_2(s')} + \text{Im}\alpha_2(s') \log(s/s') \right] \right\}, \quad (4.15)$$

$$\begin{aligned} \beta_3(s) &= \frac{g^2}{2s} \left(1 - \frac{s_2}{s}\right) \left(1 - \frac{s_3}{s}\right) \exp \left\{ \frac{1}{\pi} \int_0^\infty \frac{ds'}{s' - s} \right. \\ & \quad \times \left[\tan^{-1} \frac{\text{Im}\beta_3(s')}{\text{Re}\beta_3(s')} + \text{Im}\alpha_3(s') \log(s/s') \right] \left. \right\}, \end{aligned} \quad (4.16)$$

where s_1 , s_2 , and s_3 are given by Eq. (4.12).

The $\beta_n(s)$ are again coupled to the $\alpha_n(s)$ through the unitarity condition at $l = \alpha$. If the first N trajectories are to be coupled, the N unitarity relations are

$$\begin{aligned} 1 &= -2i(s)^{1/2} \sum_{n=1}^N \frac{\beta_n(s) \exp[-(\alpha_p^*(s) - \alpha_n(s))\xi(s)]}{\alpha_p^*(s) - \alpha_n(s)} \\ & \quad - \frac{ig^2}{(s)^{1/2}} \left[Q_{\alpha_p^*(s)}(1 + m^2/2s) \right. \\ & \quad \left. - \sum_{n=1}^N P_{n-1}(1 + m^2/2s) \frac{\exp[-(\alpha_p^*(s) + n)\xi(s)]}{\alpha_p^*(s) + n} \right] \\ & \quad p = 1, 2, \dots, N. \end{aligned} \quad (4.17)$$

We now have a system of N integral equations coming from (2.6) coupled by the N unitarity equations (4.17).

The approximate trajectories generated in this way will, of course, not in general satisfy the requirements that $\alpha_2(s_1) = -\frac{3}{2}$ and $\alpha_3(s_2) = -2$, $\alpha_3(s_3) = -\frac{5}{2}$. One could guarantee these conditions, at the expense of the correct asymptotic behavior of the α 's, by replacing the dispersion relation (2.1) with subtracted equations:

$$\begin{aligned} \alpha_2(s) &= -\frac{3}{2} + \frac{s - s_1}{\pi} \int \frac{ds'}{(s' - s_1)(s' - s)} \text{Im}\alpha_2(s'), \quad (4.18) \\ \alpha_3(s) &= -2 \left(\frac{s - s_3}{s_2 - s_3} \right) - \frac{5}{2} \left(\frac{s - s_2}{s_3 - s_2} \right) + \frac{(s - s_1)(s - s_2)}{\pi} \\ & \quad \times \int \frac{ds'}{(s' - s_1)(s' - s_2)(s' - s)} \text{Im}\alpha_3(s'). \end{aligned} \quad (4.19)$$

Whether it is more desirable to use unsubtracted equations, thus obtaining approximate solutions which behave correctly at large s , and giving up $\alpha_3(s_2) = -2$,

etc., or using subtracted equations and losing the right infinity behavior, is an open question. The values s_1, s_2, s_3 , etc., are, presumably, closer to the region of small positive s (where we would like to have the least error) than $s = \infty$, and perhaps this argues in favor of the subtracted form. On the other hand, use of the subtracted equations leads to $\alpha(s)$ that diverge as a power of s in the limit $s = \infty$, instead of merely approaching the wrong numerical value.

It should be pointed out here that unitarity is not the only condition coupling the $\beta_n(s)$ to the $\alpha_n(s)$. An equally valid, though perhaps not as fundamental condition comes from the Mandelstam symmetry, which states that

$$S(s, l) = e^{i\pi(2l+1)} S(s, -l-1). \quad (4.20)$$

Substituting the expression (3.8) for $A(s, l)$ and demanding satisfaction of (4.20) at N integers or half-integers also gives N coupling equations of a similar type as the unitarity condition.

The determination of a trajectory to any desired accuracy by coupling in a sufficient number of other trajectories would now appear to be simply a computational question. However, the equations we have written down are correct only for normal trajectories, which means trajectories for which α and b are real analytic functions of s with only a right-hand cut. It is well established^{12,13} that infinitely many trajectories do not have these properties. For instance, infinitely many trajectories meet in pairs at a negative energy \bar{s} , split into a complex conjugate pair and meet again at $s=0$ and $l=-\frac{1}{2}$, which is an accumulation point of trajectories.¹⁵ The behavior of these trajectories has been studied by Newton and Desai.¹⁶ In this case, there is a left-hand cut in s , from \bar{s} to 0. The branch point \bar{s} need not be real. In the case of a Yukawa potential with $g^2=2$ and $m=1$, the third trajectory, for example, starting at $l=-3$ for $s=-\infty$, goes continuously to $l=-5$ for $s=+\infty$, indicating a pair of branch points in the complex s plane.

When the $\alpha(s)$ and $b(s)$ have singularities other than the right-hand cut, it is necessary to extend the dispersion integrals to include the additional cuts. For such modified dispersion relations to be useful, it is, among other things, necessary to know the locations of the branch points and these are not under very good control. (There is a possibility that they could be put in as undetermined parameters and determined from self-consistency, since the two trajectories involved must become complex conjugate pairs and meet at $l=-\frac{1}{2}$ at threshold.)

It has been observed by Cheng⁴ that even though two trajectories meet, combinations such as $\alpha_1+\alpha_2$, $\alpha_1\alpha_2$ and similar combinations for the b 's remain real

analytic. It would be tempting to write the integral equations for these combinations. However, it is then no longer possible to express all the needed zeros of functions such as $b_1(s)+b_2(s)$ in terms of low moments of the potential, since these zeros do not have the simple significance of the indeterminacy points.

There is some indication that the abnormal trajectories may not be of great physical significance. In the case of a Yukawa potential with $g^2=2$, for example, the first and the second trajectories are normal, the third is not. However, a calculation by Ahmadzadeh¹⁰ indicates that $A(s, l)$ at $l=0, 1$, and 2 is accurately obtained by using the modified Khuri representation (3.9) with only the two top trajectories included. For stronger attractive potentials, more trajectories will be needed for an accurate representation of the partial-wave amplitude, but more of the top trajectories will have become normal.

V. THE RELATIVISTIC CASE AND APPLICATIONS TO BOOTSTRAP CALCULATIONS

In the preceding sections, a method for calculating Regge trajectories and their residues for a superposition of Yukawa potentials has been described. The input used to formulate the method consisted of dispersion relations for $\alpha_n(s)$ (2.1) and $b_n(s)$, unitarity (2.8), and a relation allowing calculation of the zeros of $b_n(s)$ in terms of low-order moments of the potential (Sec. IV). The method can be generalized to relativistic dynamics by reformulating each input relation in relativistic terms.

We start with the observation that generalized potentials

$$V_{\text{direct}}(s, t) = -\frac{1}{\pi} \int_m^\infty \frac{dt' \rho_t(t', s)}{t' - t} \quad (5.1)$$

and

$$V_{\text{exchange}}(s, u) = -\frac{1}{\pi} \int_{m'}^\infty \frac{du' \rho_u(u', s)}{u' - u} \quad (5.2)$$

can be defined,¹⁷ which play the same role in determining the amplitude through dispersion relations as does the nonrelativistic potential.¹⁸ We conjecture that $\alpha_n(s)$ satisfies the dispersion relation

$$\alpha_n(s) = \alpha_n(\infty) + \frac{1}{\pi} \int_{s_0}^\infty \frac{ds'}{s' - s - i\epsilon} \text{Im} \alpha_n(s') \quad (5.3)$$

and that $b_n(s)$ is again an analytic function of s with only a right-hand cut (as usual, crossing of trajectories will introduce left-hand cuts into α_n and b_n in some cases). Due to the more complicated nature of the relativistic potential, the high-energy limits of α_n and b_n are not generally known in advance. The elastic

¹⁵ V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 328 (1962).

¹⁶ B. R. Desai and R. G. Newton, Phys. Rev. **130**, 2109 (1963).

¹⁷ Henceforth, s takes on the usual relativistic significance, e.g., $s=4m^2+4q^2$ for the equal mass case instead of the nonrelativistic definition of $s=q^2$.

¹⁸ G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).

unitarity condition is changed from (2.8) only by a kinematic factor:

$$[A(s, l) - A(s, l^*)^*]/2i = [(s - 4m^2)/s]^{1/2} A(s, l) A(s, l^*)^*. \quad (5.4)$$

More generally, several channels can be coupled together, turning the unitarity condition into a matrix relation, but we shall not consider this complication. The remaining input conditions, on the zeros of b_n , were derived in Sec. IV from properties of the Schrödinger equation, which does not carry over directly to the relativistic case. Nonrelativistic scattering can equivalently be described by Fredholm theory, however, in which one can work directly with integrals over the weight function of a superposition of Yukawa potentials. Fivel¹⁹ has shown how the zeros of b_n appear in Fredholm theory, and verified that one obtains the same relations as in the Schrödinger method. His procedure can perhaps be directly generalized to relativistic dynamics, using the energy-dependent Yukawa weight functions of Eqs. (5.1) and (5.2).

Thus, the procedure of the preceding sections for computing trajectories and their residues from a given potential can be followed exactly, with the obvious changes due to the different kinematic factor in the unitarity relation. The exchange potential (5.2) makes it necessary to consider even and odd signatures²⁰ separately, thus reducing the problem to one where the singularities in the potential occur only at positive t .

In our previous discussion, the details of the potential could be introduced either through the zeros of b_n , or by explicitly adding the potential, or by both methods. We shall continue to use both methods, but since the relativistic potential is a complicated energy-dependent function it seems best to place our main reliance on including it explicitly in the amplitude. In other words, the description of the potential through zeros of b_n probably converges more slowly when the potential is energy-dependent.

In potential theory, we also had the option, for trajectories starting to the left of $l = -1$, of fixing $\alpha_n(s)$ at known indeterminacy points such as $l = -\frac{3}{2}$, or fixing $\alpha_n(\infty)$. Relativistically, we do not know the asymptotic behavior of $\alpha_n(s)$ or $b_n(s)$ in advance, and this increases the motivation for fixing α_n at known indeterminacy points whenever possible.

Now the relativistic potential can be described in terms of scattering in crossed channels; in particular, it can be related to the leading Regge trajectories in the crossed channels. When this information is inserted into the potential, we have a bootstrap calculation, where the Regge poles parameters computed in the s channel are required to be consistent with those in the crossed channels.

The description of the potential in terms of leading Regge trajectories has been given by Chew,³ and we can simply take over his results. It is convenient to retain only the fairly long-range parts of the potential, and $\alpha_n(s)$ and $b_n(s)$ at fairly low s (strip approximation), since we cannot calculate accurately the very short-range potential or the asymptotic behavior of $\alpha_n(s)$ and $b_n(s)$ anyway. Use of the strip approximation also reduces the amount of doublecounting when the potential and Regge terms in the s channel are added together to form the amplitude; this is not a serious problem in any case because each time a Regge pole is added in the s channel, our procedure calls for subtracting the corresponding pole out of the potential, as in Eq. (3.9).

One final comment concerns the convergence of the method. At $t > 0$, leading trajectories in the potential may rise to $J(t) > 1$, creating the possibility of divergences. At $t \leq 0$, however, all trajectories must remain within the Froissart bound²¹ $J(t) \leq 1$. It is thus desirable to express the partial-wave amplitudes $A^\pm(s, l)$ for signatures \pm in terms of an integral over $t \leq 0$ instead of the usual integration over A_t^\pm at $t > 0$. The appropriate formula has been given by Chew,³ following a suggestion of Wong²²;

$$A^\pm(s, l) = \frac{1}{2} \int_{-1}^1 dz P_l(z) A^\pm(s, z) - \frac{\sin \pi l}{\pi} \int_{-\infty}^{-1} dz Q_l(-z) A^\pm(s, z). \quad (5.5)$$

VI. DISCUSSION

Our proposal for making bootstrap calculations in terms of Regge parameters is rather closely related to the "generalized strip approximation" of Chew.³ The difference is a technical one; we work directly with the Regge parameters whereas Chew's approach involves the N/D method. A possible drawback of our proposal relative to Chew's is that we are involved with nonlinear equations even at the level of potential theory, where the N/D method is still linear.

Both of these methods have certain deficiencies in common. The asymptotic behavior of $\alpha_n(s)$ and $b_n(s)$ is not known as $s \rightarrow \infty$ in the relativistic case; this may not be of great importance because we are primarily interested in these parameters at small s . Even at small s , the usual Khuri factor gives correct threshold behavior for $A(s, l) \sim q_s^{2l}$, but not for $\text{Im} A(s, l) \sim q_s^{4l+1}$. This is directly related to the fact that the Khuri factor implies a double spectral function with boundary $q_s^2 = 0$, $t = 4u^2$, instead of the correct curved boundary. It is also presumably related to neglect of the infinite set of trajectories that converge on $l = -\frac{1}{2}$ at threshold. As discussed at the end of Sec. IV, there is some reason

¹⁹ D. Fivel (private communication).

²⁰ S. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962); and E. Squires, Nuovo Cimento **25**, 242 (1962).

²¹ M. Froissart, Phys. Rev. **123**, 1053 (1961).

²² D. Wong (private communication to G. F. Chew, 1962).

to hope that these neglected trajectories are not of great physical significance except in the immediate vicinity of threshold.

Independently of our work, Mandelstam and Sharp²³ have also generalized the one-pole equations described in Sec. II. They continue to consider only one pole, adding the relativistic potential and treating the Khuri ξ in a rather different manner.

Numerical calculations based on the potential theory relations of Secs. II and III are in progress. It is hoped

that comparison with the exact results of Ahmadzadeh *et al.*⁹ will give an idea of how rapidly the addition of more Regge poles converges to the full potential theory answer.

ACKNOWLEDGMENTS

The early stages of this work were aided by helpful discussions with Professor Murray Gell-Mann, for which we are very grateful. Two of the authors (S.F. and P.K.) wish to acknowledge the hospitality of the Physics Division of the Aspen Institute for Humanistic Studies, where much of the work was carried out.

²³ S. Mandelstam and D. Sharp (private communication).